University of California, Berkeley
Physics H7C Fall 2002 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION II

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for two $8\frac{1}{2} \times 11$ inch sheets containing any information you wish on both sides. Calculators are not needed, but you may use one if you wish. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points)
A nonrelativistic particle of mass m, moving in one dimension x, is confined by infinite potential walls at $x = 0$ and $x = L$. *It is in the ground state.* (All of your answers to this problem should depend at most on L, m, and fundamental constants.)

(**a**) (10 points)
Write down the particle's *normalized, time-dependent* ground-state wavefunction $\psi_1(x,t)$.

Solution:
This is an infinite-square-well problem. Taking $V = 0$ at the bottom of the well, the eigenfunctions $u_E(x)$ of the time-independent Schroedinger equation (TISE)

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}u_n(x) = E_n u_n(x) \ ,$$

with boundary conditions $u_n(0) = u_n(L) = 0$, are proportional to

$$u_n(x) \propto \sin\frac{n\pi x}{L} \ ,$$

where n is an integer ≥ 1. Normalizing the eigenfunctions

$$1 = \int_0^L u_n^* u_n \, dx$$

requires the constant of proportionality to be equal to $\sqrt{\frac{2}{L}}$. From the TISE the eigenvalues E_n are equal to

$$E_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2} \ .$$

Here the particle is in the ground state $n = 1$. Since this is a state of definite energy, the solution to the time-dependent Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

is

$$\psi(x,t) = u_1(x)\exp\left(-i\frac{E}{\hbar}t\right) \ .$$

Therefore

$$\psi(x,t) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\exp\left(-i\frac{\hbar}{2m}\frac{\pi^2}{L^2}t\right) \ .$$

Full credit is received for simply writing down this solution.

(**b**) (10 points)
The uncertainty Δp in p, the particle's momentum, is defined by

$$\Delta p \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2} \ ,$$

where $\langle \ \rangle$ denotes an expectation value. Using either a cogent argument or a calculation, provide an *order-of-magnitude estimate* for Δp.

Solution:
Continuing to take $V = 0$ at the bottom of the well, the operator p^2 is equal to $2m\mathcal{H}$, where $\mathcal{H}$, the Hamiltonian, is the operator on the left-hand side of either Schoedinger equation. Here, since the particle is in a state of definite energy E, $\langle\mathcal{H}\rangle = E$, and

$$\begin{aligned}\langle p^2\rangle &= 2mE \\ &= 2m\frac{\hbar^2}{2m}\frac{\pi^2}{L^2} \\ &= \hbar^2\frac{\pi^2}{L^2} \ .\end{aligned}$$

When it operates on u_1, the operator $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$ yields a single term proportional to $\cos\frac{\pi x}{L}$, which is odd about the center of the well. Since u_1^* is an even function about the same point, the integral

$$\langle p\rangle = \int_0^L u_1^* \frac{\hbar}{i}\frac{\partial}{\partial x} u_1 \, dx$$

vanishes. Therefore $\langle p\rangle = 0$ (this also may be argued to be true by symmetry). Thus

$$\Delta p \equiv \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\frac{\pi}{L} \ .$$

This is an exact result. However the problem requires only an order-of-magnitude estimate of Δp. For this, one may alternatively use the Heisenberg uncertainty principle

$$\Delta p \Delta x \geq \tfrac{\hbar}{2} \ ,$$

where

$$\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2} \ .$$

Estimating $\Delta x \approx \frac{L}{2}$, one has

$$\begin{aligned}\Delta p &\geq (\approx \frac{\hbar}{L}) \\ &\approx \frac{\hbar}{L} \ ,\end{aligned}$$

which has the same order of magnitude as the exact result, and likewise receives full credit.

(**c**) (15 points)

The uncertainty $\Delta(p^2)$ in p^2, the square of the particle's momentum, is defined by

$$\Delta(p^2) \equiv \sqrt{\langle p^4\rangle - \langle p^2\rangle^2} \ .$$

Using either a cogent argument or a calculation, provide an *exact value* for $\Delta(p^2)$.

Solution:

The solution to part (**b**) found that u_1 is an eigenfunction of p^2 with eigenvalue $\hbar^2\pi^2/L^2$. Since the operator p^4 is equivalent to the operator p^2 operating twice, u_1 is also an eigenfunction of p^4 with eigenvalue $\hbar^4\pi^4/L^4$. Therefore

$$\Delta(p^2) \equiv \sqrt{\langle p^4\rangle - \langle p^2\rangle^2} = 0 \ .$$

Problem 2. (30 points)

A particle of mass m that moves in a one-dimensional harmonic oscillator potential

$$V(x) = \tfrac{1}{2} m\omega_0^2 x^2$$

has orthonormal bound states with exact energies, relative to the bottom of the well, equal to

$$E_n = \hbar\omega_0(n + \tfrac{1}{2}) \ ,$$

where n is an integer ≥ 0. This particle interacts not only with the potential walls, but also with a thermal bath at temperature T, with which it is free to exchange energy and with which it is in thermal equilibrium. Because of the bath, the particle's probability of occupying state n is proportional to the Boltzmann factor $\exp(-E_n/k_BT)$, where k_B is Boltzmann's constant.

Calculate the particle's energy expectation value $\langle E\rangle$ as a function of ω_0, T, and fundamental constants. [*Hint*: As $T \to 0$ and $T \to \infty$, your result should approach simple and sensible limiting values.]

Solution:

In general the particle's wavefunction u can be expressed as a linear combination (with coefficients a_n) of the energy eigenfunctions u_n, which form a basis set:

$$u = \sum_{n=0}^{\infty} a_n u_n \ .$$

Therefore the probability P_n of occupying state n is

$$\begin{aligned}P_n &= \int_{-\infty}^{\infty} a_n^* u_n^* \, a_n u_n \, dx \\ &= a_n^* a_n \ ,\end{aligned}$$

using the normality of the u_n. According to the problem, P_n has a Boltzmann energy dependence

$$P_n \propto \exp(-E_n/k_BT) \ ,$$

where

$$E_n = \hbar\omega_0(n + \tfrac{1}{2}) \ .$$

Therefore, substituting $\beta \equiv (1/k_BT)$,

$$a_n^* a_n = C\exp(-n\beta\hbar\omega_0) \ ,$$

where C is a constant of proportionality. Using the orthonormality of the u_n, C is determined by the requirement

$$\begin{aligned}
1 &= \langle u^* u \rangle \\
&= \sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} a_m^* u_m^* \, a_n u_n \, dx \\
&= \sum_{n=0}^{\infty} a_n^* \, a_n \\
&= C \sum_{n=0}^{\infty} \exp\left(-n\beta\hbar\omega_0\right) \\
&= \frac{C}{1 - \exp\left(-\beta\hbar\omega_0\right)} \\
C &= 1 - \exp\left(-\beta\hbar\omega_0\right) .
\end{aligned}$$

The energy expectation value is

$$\langle E \rangle = \int_{-\infty}^{\infty} u^* \mathcal{H} u \, dx ,$$

where $\mathcal{H} u_n = E_n u_n$. Again using the orthonormality of the u_n,

$$\begin{aligned}
\langle E - \tfrac{1}{2}\hbar\omega_0 \rangle &= \sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} a_m^* u_m^* \, n\hbar\omega_0 \, a_n u_n \, dx \\
&= \sum_{n=0}^{\infty} a_n^* \, n\hbar\omega_0 \, a_n \\
&= C \sum_{n=0}^{\infty} n\hbar\omega_0 \exp\left(-n\beta\hbar\omega_0\right) \\
&= \big(1 - \exp\left(-\beta\hbar\omega_0\right)\big) \frac{-\partial}{\partial\beta} \sum_{n=0}^{\infty} \exp\left(-n\beta\hbar\omega_0\right) \\
&= \big(1 - \exp\left(-\beta\hbar\omega_0\right)\big) \frac{-\partial}{\partial\beta} \frac{1}{1 - \exp\left(-\beta\hbar\omega_0\right)} \\
&= \big(1 - \exp\left(-\beta\hbar\omega_0\right)\big) \frac{\hbar\omega_0 \exp\left(-\beta\hbar\omega_0\right)}{\big(1 - \exp\left(-\beta\hbar\omega_0\right)\big)^2} \\
&= \frac{\hbar\omega_0 \exp\left(-\beta\hbar\omega_0\right)}{1 - \exp\left(-\beta\hbar\omega_0\right)} \\
&= \frac{\hbar\omega_0}{\exp\left(\beta\hbar\omega_0\right) - 1} .
\end{aligned}$$

Therefore $\langle E \rangle$ reduces to $\frac{1}{2}\hbar\omega_0$ when $T \to 0$ ($\beta \to \infty$) and, Taylor expanding the denominator, to $\frac{1}{\beta} = k_B T$ when $T \to \infty$. (Full credit was given to an otherwise correct result that did not sum the geometric series.)

Problem 3. (35 points)
A mostly uniform transparent plate of refractive index $n = 2$, normally illuminated by a plane wave of *vacuum* wavelength λ_0, occupies a thin region whose downstream edge is mostly the plane $z = 0$. However, the plate has a small protrusion described by the the cylinder ($0 < z < \lambda_0/4$, $x^2 + y^2 < R^2$). In other words, within a radius R from the z axis, the plate is thicker than elsewhere by $\lambda_0/4$, where λ_0 is the *vacuum* wavelength. An observer is located on the z axis downstream of the protrusion, at $z \gg R$. You may *not* assume that Fraunhofer conditions apply. Your answers to both parts must be justified!

a. (15 points) Walking *from* very large z *toward* the plate, the observer sees the irradiance rise to a local maximum at $z = z_1$. What is z_1?
Solution:
As the observer (with diminishing coordinate z) walks toward the plate, the Fresnel zone outer radii $R_m = \sqrt{m\lambda_0 z}$ become smaller. Eventually the first Fresnel zone outer radius R_1 will become as small as the radius R of the protrusion.

The salient characteristic of Fresnel zones is that their contributions U_m to the observed optical disturbance alternate in sign. Also, the sum

$$U_{\geq 2} \equiv \sum_{m=2}^{\infty} U_m$$

is opposite in sign to U_1. Their partial cancellation causes the observed optical disturbance

$$U_{\text{obs}} \equiv \sum_{m=1}^{\infty} U_m$$

to be not as large as the single contribution U_1 from the first zone.

When $R_1 = R$, 100% of the first Fresnel zone will have a shift in phase, relative to the other zones, caused by the increased thickness of the protrusion. This phase shift will upset the partial cancellation, producing a larger U_{obs}. Therefore the first maximum will occur at z_1 such that

$$\begin{aligned}\sqrt{\lambda_0 z_1} &= R \\ z_1 &= \frac{R^2}{\lambda_0} \ .\end{aligned}$$

b. (20 points) At $z = z_1$, what is the ratio of the observed irradiance to the irradiance that would be observed if the plate had no protrusion?

Solution:

The protrusion causes the plate to be thicker by $\Delta z \equiv \lambda_0/4$ within its area. Inside the plate, the wavelength λ is equal to λ_0/n, where $n = 2$ is its refractive index. Therefore, upon exiting the protrusion, the optical disturbance develops an additional phase shift

$$\begin{aligned}\Delta\phi_{\text{protrusion}} &= k\Delta z \\ &= \frac{2\pi}{\lambda_0/n}\lambda_0/4 \\ &= \pi\end{aligned}$$

relative to light entering the protrusion. At the same z, light that doesn't pass through the protrusion will develop an additional phase shift in vacuum:

$$\begin{aligned}\Delta\phi_{\text{outside protrusion}} &= k_0\Delta z \\ &= \frac{2\pi}{\lambda_0}\lambda_0/4 \\ &= \pi/2 \ .\end{aligned}$$

Therefore, when the protrusion radius is equal to that of the first Fresnel zone, U_1 will be $\pi - \pi/2 = \pi/2$ out of phase with $U_{\geq 2}$. This means that the two will add incoherently. Thus

$$\begin{aligned}I_{\text{obs}} &= I_1 + I_{\geq 2} \\ &= 4I_{\text{no}} + I_{\text{no}} \\ &= 5I_{\text{no}} \ ,\end{aligned}$$

where the subscript $_{\text{no}}$ refers to the total that would be observed if the protrusion did not exist. Here we have used the standard Fresnel-zone facts that $|U_1| = 2|U_{\text{no}}|$ and $|U_{\geq 2}| = |U_{\text{no}}|$.